United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2

Thursday 30 January 2020

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Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. Write on one side of the paper only and start each question on a fresh sheet.
4. Rough work *should* be handed in, but should be clearly marked.
5. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
6. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
7. Staple all the pages neatly together in the top *left* hand corner, with questions $1, 2, 3, 4$ in order, and the cover sheet at the front.
8. To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 31 January. Candidates sitting the paper in time zones more than 3 hours ahead of GMT must sit the paper on Friday 31 January (as defined locally).
9. In early March, twenty-four students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (31 March–5 April 2020). At the training session, students sit a pair of IMO-style papers and some students will be selected for further training and selection examinations. The UK Team of six for this year's IMO (to be held in St Petersburg, Russia 8–18 July 2020) will then be chosen.
10. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

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1. A sequence $a_1, a_2, a_3, \ldots$ has $a_1 > 2$ and satisfies:

$$a_{n+1} = \frac{a_n(a_n - 1)}{2}$$

for all positive integers n. For which values of a_1 are all the terms of the sequence odd integers?

2. Describe all collections S of at least four points in the plane such that no three points are collinear and such that every triangle formed by three points in S has the same circumradius.

(The circumradius of a triangle is the radius of the circle passing through all three of its vertices.)

3. A 2019×2019 square grid is made up of 2019^2 unit cells. Each cell is coloured either black or white. A colouring is called *balanced* if, within every square subgrid made up of k^2 cells for $1 \leq k \leq 2019$, the number of black cells differs from the number of white cells by at most one. How many different balanced colourings are there?

(Two colourings are different if there is at least one cell which is black in exactly one of them.)

4. A sequence $b_1, b_2, b_3, \ldots$ of nonzero real numbers has the property that

$$b_{n+2} = \frac{b_{n+1}^2 - 1}{b_n}$$

for all positive integers n.

Suppose that $b_1 = 1$ and $b_2 = k$ where $1 < k < 2$. Show that there is some constant B, depending on k, such that $-B \leq b_n \leq B$ for all n. Also show that, for some $1 < k < 2$, there is a value of n such that $b_n > 2020$.

